BURKHOLDER-DAVIS-GUNDY TYPE INEQUALITIES OF THE ITÔ STOCHASTIC INTEGRAL WITH RESPECT TO LÉVY NOISE ON BANACH SPACES

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Abstract. The aim of this note is to give some Burkholder-Davis-Gundy type inequalities which are valid for the Ito stochastic integral with respect to Banach valued Lévy noise.

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1. Introduction

Let us assume that $(S, \mathcal{S})$ is a metric space with Borel σ algebra $\mathcal{S}$ and $\tilde{\eta}$ is a time homogeneous compensated Poisson random measure defined on a filtered probability space $(\Omega; \mathcal{F}; (\mathcal{F}_t)_{0\le t<\infty}; \mathbb{P})$ with intensity measure ν on S, to be specified later. Let us assume that $1 < p \le 2$, E is a Banach space of martingale type p, see e.g. the Appendix of [6] for a definition. We consider in the following the Itô stochastic integral $I = \{I(t), 0 \le t < \infty\}$ driven by the compensated Poisson random measure $\tilde{\eta}$, i.e.

$$I(t) \quad = \quad \int_0^t \int_S \xi(s;x)\tilde{\eta}(dx;ds),$$

where $\xi : [0,T] \times \Omega \times S \to E$ is a progressively measurable processes satisfying certain integrability conditions specified later. We are interested in Inequalities satisfied by the process I. In particular, we will show that for any $q = p^n$, $n \in \mathbb{N}$, there exist constants C and $\bar{C}$, only depending on E, p and q, such that

$$\mathbb{E} \sup_{0\le s\le t} |I(s)|^q \le C\,\mathbb{E}\left(\int_0^t \int_S |\xi(s;x)|^p\,\eta(dx;ds)\right)^{\frac{q}{p}}$$

$$\le \bar{C}\left(\mathbb{E}\int_0^t \int_S |\xi(s;x)|^q\,\nu(dx;ds) + \mathbb{E}\left(\int_0^t \int_S |\xi(s;x)|^p\,\nu(dx;ds)\right)^{\frac{q}{p}}\right).$$

From this inequalities one can derive similar inequalities for martingales of pure jump type. To be more precise, let X be a martingale, such that there exists a Lévy process L and a progressively process $h : [0,\infty) \to L(E,E)$, satisfying some integrabilities condition specified later, with

$$X(t) = \int_0^t h(s)\,dL(s), \quad t \ge 0.$$

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Then there exist constants $C > 0$ and $\bar{C} > 0$ such that

$$\begin{aligned}\mathbb{E} \sup_{0\le s\le t} |X(s)|^q &\le C\,\mathbb{E}\left(\sum_{0\le s\le t} |\Delta_s X|^p\right)^{\frac{q}{p}} \\ &\le \bar{C}\left(\mathbb{E}\sum_{0\le s\le t} |\Delta_s X|^q + \mathbb{E}\left(\sum_{0\le s\le t} \mathbb{E}[\,|\Delta_s X|^p\,|\mathcal{F}_{s-}]\right)^{\frac{q}{p}}\right),\end{aligned}$$

where $\Delta_t X = X(t) - X(t-)$, $t > 0$.

Notation . *Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\bar{\mathbb{N}} := \mathbb{N}_0 \cup \{\infty\}$. By $\mathcal{M}_I(S \times \mathbb{R}_+)$ we denote the family of all $\bar{\mathbb{N}}$-valued measures on $(S \times \mathbb{R}_+, \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+))$. By $\mathcal{M}^+(S)$ we denote the set of all positive measures on S. For any Banach space Y and number $q \in [1, \infty)$, we denote by $\mathcal{N}(\mathbb{R}_+; Y)$ the space of (equivalence classes) of progressively-measurable processes $\xi : \mathbb{R}_+ \times \Omega \to Y$ and by $\mathcal{M}^p(\mathbb{R}_+; Y)$ the Banach space consisting of those $\xi \in \mathcal{N}(\mathbb{R}_+; Y)$ for which $\mathbb{E}\int_0^\infty |\xi(t)|_Y^p\,dt < \infty$.*

2. Main results

Let us first introduce the notation of time homogeneous Poisson random measures over a filtered probability space.

Definition 2.1. *Let $(S, \mathcal{S})$ be a measurable space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*
A Poisson random measure η on $(S, \mathcal{S})$ over $(\Omega, \mathcal{F}, \mathbb{P})$, is a measurable function $\eta : (\Omega, \mathcal{F}) \to (\mathcal{M}_I(S \times \mathbb{R}_+), \mathcal{B}(\mathcal{M}_I(S \times \mathbb{R}_+))$, where $\mathcal{B}(\mathcal{M}_I(S \times \mathbb{R}_+)$ is the σ-field on $\mathcal{M}_I(S \times \mathbb{R}_+)$ generated by functions $i_B : \mathcal{M}_I(S \times \mathbb{R}_+) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$, such that

(i) *η is independently scattered, i.e. if the sets $B_j \in \mathcal{S} \times \mathcal{B}(\mathbb{R}_+)$, $j = 1, \cdots, n$ are pairwise disjoint, then the random variables $\eta(B_j)$, $j = 1, \cdots, n$ are pairwise independent.*
(ii) *for each $B \in \mathcal{S}$, $\eta(B) := i_B \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter*[1] *$\mathbb{E}\eta(B)$.*
(iii) *for each $U \in \mathcal{S}$, the $\bar{\mathbb{N}}$-valued processes $(N(t, U))_{t>0}$ defined by*

$$N(t, U) := \eta((0, t] \times U), \quad t > 0$$

is $(\mathcal{F}_t)_{t\ge 0}$-adapted and its increments are independent of the past, i.e. if $t > s \ge 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathcal{F}_s$.

Poisson random measures arise in a natural way. E.g. by means of a Lévy process one can construct a Poisson random measure.

Definition 2.2. *Let E be a Banach space. A stochastic process $\{X(t) : t \ge 0\}$ is a Lévy process if the following conditions are satisfied.*

- *for any choice $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \cdots t_n$, the random variables $X(t_0)$, $X(t_1) - X(t_0)$, …, $X(t_n) - X(t_{n-1})$ are independent;*
- *$X_0 = 0$ a.s.;*
- *For all $0 \le s < t$, the distribution of $X(t + s) - X(s)$ does not depend on s;*
- *X is stochastically continuous;*
- *the trajectories of X are a.s. cádlág on E.*

[1] If $\mathbb{E}\eta(B) = \infty$, then obviously $\eta(B) = \infty$ a.s..

The characteristic function of a Lévy process is uniquely defined and is given by the Lévy-Khinchin formula. formula. Here, we assume for simplicity that E is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. For discussion on Banach spaces we refer e.g. to [1, 3, 16]. Then, for any E-valued Lévy process $\{L(t) : t \geq 0\}$ there exists a trace class operator Q, a non negative measure ν concentrated at $E \setminus \{0\}$ and an element $m \in E$ such that

$$\mathbb{E}e^{i\langle\theta,L(t)\rangle} = \exp\Big(i\langle m,x\rangle + \frac{1}{2}\langle Qx,x\rangle + \int_E \Big(1 - e^{i\langle x,y\rangle} + 1_{(-1,1)}(|y|_E)i\langle x,y\rangle\Big)\,\nu(dy)\Big).$$

We call the measure ν *characteristic measure* of the Lévy process $\{L(t) : t \geq 0\}$. Moreover, note that the triplet (Q,m,ν) uniquely determines the law of the Lévy process. Now, one can construct a Poisson random measure with intensity measure ν.

Example 2.3. *Given a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ and a Hilbert space E. To each time homogeneous E-valued Lévy process $\{L(t) : t \geq 0\}$ on $(E,\mathcal{B}(E))$ with characteristic measure ν, we can associate a counting measure, denoted by η_L over $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ by*

$$\mathcal{B}(E)\times\mathcal{B}(\mathbb{R}_+) \ni (B,I) \mapsto \eta_L(B\times I) := \#\{s \in I \mid L_s - L_{s-} \in B\} \in \mathbb{N}_0 \cup \{\infty\}.$$

The counting measure is a time homogeneous Poisson random measure with intensity measure ν. Moreover,

$$L(t) = \int_0^t z\,\eta_L(dz,ds), \quad t \geq 0.$$

For more details on the relationship between Poisson random measure and Lévy processes we refer to Applebaum [2] Ikeda and Watanabe [12] or Peszat and Zabczyk [17].

Let us assume that $1 < p \leq 2$ and E is a Banach space of martingale type p, see e.g. the Appendix of [6] for a definition. Let us assume that $(S,\mathcal{S})$ is a measurable space and $\nu \in M^+(S)$. Suppose that $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ is a filtered probability space, $\eta : \mathcal{S}\times\mathcal{B}(\mathbb{R}_+) \to \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure with intensity measure ν defined over $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. We will denote by $\tilde\eta = \eta - \gamma$ the compensated Poisson random measure associated to η, where the compensator γ is defined by

$$\mathcal{B}(\mathbb{R}_+)\times\mathcal{S} \ni (A,I) \mapsto \gamma(A,I) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$

We have proved in [6] that there exists a unique continuous linear operator which associates with each progressively measurable process $\xi \in \mathcal{M}^p(\mathbb{R}_+;L^p(S,\nu;E))$ an adapted càdlàg E-valued process, denoted by $\int_0^t\int_S \xi(r,x)\tilde\eta(dx,dr)$, $t \geq 0$, such that if a process $\xi \in \mathcal{M}(\mathbb{R}_+,L^p(S,\nu;E))$ is a random step process with representation

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1},t_j]}(r)\,\xi_j, \quad r \geq 0,$$

where $\{t_0 = 0 < t_1 < \ldots < t_n < \infty\}$ is a finite partition of $[0,\infty)$ and for all j, ξ_j is an E-valued $\mathcal{F}_{t_{j-1}}$ measurable, p-summable random variable, then

$$\int_0^t\int_S \xi(r,x)\,\tilde\eta(dx,dr) = \sum_{j=1}^n \int_S \xi_j(x)\,\tilde\eta\,(dx,(t_{j-1}\wedge t, t_j\wedge t])\,. \tag{2.1}$$

The continuity mentioned above means that there exists a constant $C = C(E)$ independent of ξ such that

$$\mathbb{E}|\int_0^t \int_S \xi(r,x)\,\tilde\eta(dx,dr)|^p \le C\,\mathbb{E}\int_0^t \int_S |\xi(r,x)|^p\,\nu(dx)\,dr,\ t\ge 0. \tag{2.2}$$

As mentioned above, we are interested in inequalities satisfied by the stochastic integral processes given by

$$\mathbb{R}_+ \ni t \mapsto I(t) = \int_0^t \int_S \xi(s,z)\,\tilde\eta(dz,ds).$$

Proposition 2.4. *Let $1 < p \le 2$ and let E be a separable Banach space of martingale type p. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ be a filtered probability space. Assume that $\tilde\eta$ is a compensated time homogeneous Poisson random measure over $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ with intensity ν and compensator γ. Assume that $\xi \in \mathcal{M}^p(\mathbb{R}_+;L^p(S,\nu;E))$. Then*

(i) *there exists a constant $C = C_p(E)\,2^{2-p}$ only depending on E and p such that*

$$\mathbb{E}\sup_{0<t\le T}\left|\int_0^t \int_S \xi(\sigma,z)\ \tilde\eta(dz;d\sigma)\right|^r \le$$
$$C\ \left(\int_0^T \int_S \mathbb{E}\,|\xi(s,z)|^p\ \nu(dz)\ ds\right)^{\frac{r}{p}},\quad 0<r\le p.$$

(ii) *there exists a constant $C = C_p(E)\,2^{r(2+\frac1p)}(m_0-1)^{(p-1)\frac{r}{p}}$, $m_0 = \inf\{n\ge1 : p - n\frac{p}{r} \le 1\}$, only depending on E, p, and r such that*

$$\mathbb{E}\sup_{0<t\le T}\left|\int_0^t \int_S \xi(\sigma,z)\ \tilde\eta(dz;d\sigma)\right|^r \le$$
$$C\,\mathbb{E}\left(\int_0^T \int_S |\xi(s,z)|^p\ \eta(dz;ds)\right)^{\frac{r}{p}},\quad p\le r<\infty,$$

(iii) *Let q be a natural number with $q = p^n$ for a number $n\in\mathbb{N}$. If in addition*

$$\int_0^t \int_S \mathbb{E}|\xi(s,z)|^q\ \nu(dz)\ ds < \infty,$$

then

$$\mathbb{E}\sup_{0<s\le t}\left|\int_0^t \int_S \xi(s,z)\tilde\eta(dz;ds)\right|^q \le$$
$$2^{2-p}\ \sum_{l=1}^n \bar C(l)\mathbb{E}\left(\int_0^t \int_S |\xi(s,z)|^{p^l}\ \nu(dz)\ ds\right)^{p^{n-l}}.$$

where $\bar C(0) = 1$ and $\bar C(i) = \bar C(i-1)2^{p^{n-i+1}+2p^{n-1}}\,(m(n-i)-1)^{(p-1)\frac{r}{p}}$, $m(i) = [p^{i-1}(p-1)]+1$.

Remark 2.5. *If the underlying Banach space E is a Hilbert space or $\mathbb{R}^d$ equipped by the Euclidean norm, then E is a Banach space of M-type 2, and $C_2(E) = 1$. For other cases we refer to the book of Linde [16].*

Assume for the next paragraph, that E is a Hilbert space and that the time homogeneous Poisson random measure is the counting measure of the Lévy process described in Example 2.3. But before we look at the formulation of Proposition 2.4 in terms of Lévy processes, we introduce a important process associated to a Lévy process. The *jump process* $\Delta X = \{\Delta_t X,\, 0\le t<\infty\}$ of a process X is given by

$$\Delta_t X(t) := X(t) - X(t-) = X(t) - \lim_{\epsilon\to 0} X(t-\epsilon), \quad t\ge 0.$$

Assume that X arises by stochastic integration of a Lévy process of pure jump type. In particular, we assume that there exists a Lévy process L and a cádlág process $h\in\mathcal{M}^p(\mathbb{R}_+, L(Z,E))$ such that

$$X(t) := \int_0^t h(s)\, dL(s), \quad t\ge 0.$$

Then, $\Delta_t X = h(t)\Delta_t L$, $t\ge 0$. Now, the Proposition 2.4 reads as follows.

Corollary 2.6. *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P})$ be a filtered probability space and E is a Hilbert space. Let $L=\{L(t),\, 0\le t<\infty\}$ be time homogeneous E-valued Lévy process with characteristic measure ν over $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P})$, let $h\in\mathcal{M}^p(\mathbb{R}_+, L(E,\nu;E))$ be a cádlág process such that $h:[0,\infty)\to L(E,E)$ and $X=\{X(t),\, 0\le t<\infty\}$ be given by*

$$X(t) := \int_0^t h(s)\, dL(s), \quad t\ge 0.$$

Then

(ii) *there exists a constant $C = C_p(E)\, 2^{r(2+\frac1p)}(m_0-1)^{(p-1)\frac rp}$, $m_0 = \inf\{n\ge 1 : p - n\frac pr \le 1\}$, only depending on E, p, and r such that*

$$\mathbb{E}\sup_{0<t\le T}|X(t)|^r \le C\,\mathbb{E}\left(\sum_{s\le t}|\Delta_s X|^p\right)^{\frac rp}, \quad p\le r<\infty,$$

(iii) *Let q be a natural number with $q=p^n$ for a number $n\in\mathbb{N}$. If in addition*

$$\mathbb{E}\sum_{s\le t}|\Delta_s X|^q < \infty,$$

then

$$\mathbb{E}\sup_{0<s\le t}|X(t)|^q \;\le\; 2^{2-p}\sum_{l=1}^n \bar{C}(l)\mathbb{E}\left(\sum_{s\le t}\mathbb{E}\left[|\Delta_s X|^{p^l}\mid \mathcal{F}_{s-}\right]\right)^{p^{n-l}}.$$

where $\bar C(0)=1$ and $\bar C(i) = \bar C(i-1)2^{p^{n-i+1}+2p^{n-1}}(m(n-i)-1)^{(p-1)\frac rp}$, $m(i) = [p^{i-1}(p-1)]+1$.

3. Proof of the Inequalities in Proposition 2.4

The proof of Inequality (i) is taken from Corollary C.2 of Brzeźniak and Hausenblas [6]. If $r=p$ Inequality (ii) follows by the definition of the compensator. Hence, we give here a proof valid for $r\ge p$. Inequality (iii) can be shown by induction on n and can be found also in [19] or [4].

Proof of Inequality (i): Before beginning let us state the following Lemma. The proof of this Lemma can be found by direct calculation or in [6, Lemma C.2].

Lemma 3.1. *Suppose that* $\xi \sim$ *Poiss* (λ)*, where* $\lambda > 0$*. Then, for all* $p \in [1,2]$*,*

$$\mathbb{E}|\xi-\lambda|^p \le 2^{2-p}\lambda.$$

In the proof of Inequality (i), we will approximate ξ by a sequence of simple functions, i.e.

$$\int_S \xi(s,z)\tilde\eta(dz,ds) = \lim_{n\to\infty}\int_S \xi_n(s,z)\tilde\eta(dz,ds),$$

where $\xi_n \to \xi$ in $\mathcal{M}^p(\mathbb{R}_+;L^p(\nu,E))$.

Therefore, we first show that the inequality is valid for a simple function, and then extend the inequality to the completion of the set of simple functions, i.e. to all progressively measurable functions which are L^p-integrable. Thus, we suppose here and hereafter that ξ is a simple function. In particular, we suppose that ξ has the following representation

$$\xi = \sum_{k=1}^K 1_{(s_{k-1},s_k]}(s)\sum_j^J\sum_i^I \xi_{ji}^k\, 1_{A_{ji}^k\times B_j^k}$$

with $s_k - s_{k-1} = \tau > 0$, $\xi_{ji}^k \in E$, $A_{ji}^k \in \mathcal{F}_{s_{k-1}}$, $k=1,\ldots,K$, and $B_j \in \mathcal{B}(S_\epsilon)$, the finite families of sets $(A_{ji}\times B_j)$ and (B_j) being pair-wise disjoint and $\nu(B_j)\le 1$. Let us notice that

$$\int_0^T\int_S \xi(s,x)\tilde\eta(dx,ds) = \sum_k^K\sum_j^J\left(\sum_i^I 1_{A_{ji}^k}\,\xi_{ji}^k\right)\tilde\eta(B_j\times(s_{k-1},s_k]).$$

The Burkholder-Davis-Gundy inequality, see Inequality A.2, gives

$$\mathbb{E}\left|\int_0^T\int_S \xi(s,x)\tilde\eta(dx,ds)\right|^p \le$$
$$C_p(E)\,\mathbb{E}\left(\sum_k^K\sum_j^J \mathbb{E}\left|\sum_i^I 1_{A_{ji}^k}\,\xi_{ji}^k\,\tilde\eta(B_j\times(s_{k-1},s_k])\right|^p\right).$$

Recall that for fixed k, the family $\{A_{ji}^k, i=1,\ldots\}$ consists of disjoint sets. This implicates that for fixed $\omega\in\Omega$ only one term of the sum over the index i will be not equal to zero. Therefore, we can write

$$\mathbb{E}\left|\int_0^T\int_S \xi(s,x)\tilde\eta(dx,ds)\right|^p \le$$
$$C_p(E)\sum_k^K\sum_j^J\sum_i^I \mathbb{E}\left|1_{A_{ji}^k}\,\xi_{ji}^k\,\tilde\eta(B_j\times(s_{k-1},s_k])\right|^p. \tag{3.1}$$

In the next step we the fact that $\eta(B_j\times(s_{k-1},s_k])$ is Poisson distributed with parameter $\nu(B_j)\tau$. Therefore, (3.1) reads

$$C_p(E)\,\mathbb{E}\sum_k^K\sum_j^J\sum_i^I 1_{A_{ji}^k}\left|\,\xi_{ji}^k\mathbb{E}\left[\left(\sum_{l=1}^\infty 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, l-\lambda_j\right)\right|^p \mid \mathcal{F}_{k\tau}\right]$$

(note $\mathcal{F}_{k\tau}=\mathcal{F}_t$ for $t=k\tau$) and Lemma 3.1 gives

$$\mathbb{E}\left|\int_0^T\int_S \xi(s,x)\tilde\eta(dx,ds)\right|^p \le C_p(E)\;2^{2-p}\,\mathbb{E}\sum_k^K\sum_j^J\sum_i^I 1_{A_{ji}^k}\left|\,\xi_{ji}^k\right|^p\nu(B_j)\tau.$$

Going back we arrive at

(3.2)
$$\mathbb{E}\left|\int_0^T \int_S \xi(s,x)\tilde{\eta}(dx,ds)\right|^p \le C_p(E)\, 2^{2-p}\,\mathbb{E}\int_0^T \int_S |\xi(s,z)|^p\,\nu(dz)\,ds.$$

Now, assume that ξ is a progressively measurable process such that

$$\mathbb{E}\int_0^\infty \int_S |\xi(s,z)|^p \nu(dz)\,ds < \infty.$$

Due to Lemma 1.1, Chapter 1 in [9], there exists a sequence of simple functions $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to \xi$ in $\mathcal{M}^p(\mathbb{R}_+; L^p(\nu,E))$. Now, due to the definition of the stochastic integral we have

$$\mathbb{E}\left|\int_0^T \int_S \xi(s,x)\tilde{\eta}(dx,ds)\right|^p = \mathbb{E}\left|\lim_{n\to\infty}\int_0^T \int_S \xi_n(s,x)\tilde{\eta}(dx,ds)\right|^p.$$

The continuity of the norm and inequality (3.2) imply

$$\begin{aligned}\mathbb{E}\left|\lim_{n\to\infty}\int_0^T \int_S \xi_n(s,x)\tilde{\eta}(dx,ds)\right|^p &= \lim_{n\to\infty}\mathbb{E}\left|\int_0^T \int_S \xi_n(s,x)\tilde{\eta}(dx,ds)\right|^p \\ &= \lim_{n\to\infty} C_p(E)\, 2^{2-p}\,\mathbb{E}\int_0^T \int_S |\xi_n(s,z)|^p\,\nu(dz)\,ds \\ &= C_p(E)\, 2^{2-p}\,\mathbb{E}\int_0^T \int_S |\xi(s,z)|^p\,\nu(dz)\,ds.\end{aligned}$$

□

Proof of Inequality (ii). The integrals in inequality (ii) will be approximated first by the omitting the small jumps, i.e. by the following limits

$$\int_0^t \int_S \xi(s,z)\tilde{\eta}(dz,ds) = \lim_{\epsilon\to 0}\int_{S^\epsilon} \xi(s,z)\tilde{\eta}(dz,ds)$$

and

$$\int_0^t \int_S |\xi(s,z)|^p \eta(dz,ds) = \lim_{\epsilon\to 0}\int_{S^\epsilon} |\xi(s,z)|^p \eta(dz,ds)$$

where $S_\epsilon := S\setminus B_S(\epsilon)^2$. Secondly the integrand will be approximated by a sequence of simple functions, i.e.

$$\int_{S^\epsilon} \xi(s,z)\tilde{\eta}(dz,ds) = \lim_{n\to\infty}\int_{S^\epsilon} \xi_n(s,z)\tilde{\eta}(dz,ds),$$

where $\xi_n \to \xi$ in $\mathcal{M}^r(\mathbb{R}_+; L^r(\nu,E))$.

Before starting with the proof, let here and hereafter $\epsilon > 0$ be fixed. Also, we suppose here and hereafter that ξ is a simple function. In particular, we suppose that ξ has the following representation

$$\xi = \sum_{k=1}^K 1_{(s_{k-1},s_k]}(s)\sum_j^J \sum_i^I \xi_{ji}^k 1_{A_{ji}^k\times B_j^k}$$

[2] $B_S(y) := \{x\in S, |x|\le y\}$

with $s_k - s_{k-1} = \tau > 0$, $\xi_{ji}^k \in E$, $A_{ji}^k \in \mathcal{F}_{s_{k-1}}$, $k = 1, \dots, K$, and $B_j \in \mathcal{B}(S_\epsilon)$, the finite families of sets $(A_{ji} \times B_j)$ and (B_j) being pair-wise disjoint and $\nu(B_j) \le 1$. Let us notice that

$$\int_0^T \int_{S_\epsilon} \xi(s,x)\tilde{\eta}(dx,ds) = \sum_k^K \sum_j^J \left(\sum_i^I 1_{A_{ji}^k} \xi_{ji}^k \right) \tilde{\eta}(B_j \times (s_{k-1}, s_k]).$$

Let $r \ge p$, with $r = p^m$, $m \in \mathbb{N}$, be fixed. The Burkholder-Davis-Gundy inequality, i.e. inequality A.5 with $\Phi(x) = x^r$, $x \ge 0$, gives

$$\mathbb{E} \left| \int_0^T \int_{S_\epsilon} \xi(s,x)\tilde{\eta}(dx,ds) \right|^r \le$$
$$C_p(E)\, \mathbb{E} \left(\sum_k^K \sum_j^J \left| \sum_i^I 1_{A_{ji}^k} \xi_{ji}^k \tilde{\eta}(B_j \times (s_{k-1}, s_k]) \right|^p \right)^{\frac{r}{p}}.$$

Recall that for fixed k and j, A_{ji}^k, $i = 1, \dots, I$ are disjoint sets. This implicates that only one term of the inner sum will be not equal to zero. Therefore, we can write

$$\mathbb{E} \left| \int_0^T \int_{S_\epsilon} \xi(s,x)\tilde{\eta}(dx,ds) \right|^r \le$$
$$C_p(E)\, \mathbb{E} \left(\sum_k^K \sum_j^J \sum_i^I \left| 1_{A_{ji}^k} \xi_{ji}^k \tilde{\eta}(B_j \times (s_{k-1}, s_k]) \right|^p \right)^{\frac{r}{p}}. \tag{3.3}$$

Plugging in the definition of $\tilde{\eta}(B_j \times (s_{k-1}, s_k])$, the RHS of (3.3) reads

$$C_p(E)\, \mathbb{E} \left(\sum_k^K \sum_j^J \sum_i^I \left| 1_{A_{ji}^k}\, \xi_{ji}^k \left(\sum_{l=1}^\infty 1_{\{\eta(B_j \times (s_{k-1}, s_k]) = l\}}\, l - \nu(B_j)\tau \right) \right|^p \right)^{\frac{r}{p}}.$$

Using $|x-y|^p \le 2^p(|x|^p + |y|^p)$ we obtain

$$\begin{aligned}
&\dots \le C_p(E)\, 2^r \\
&\mathbb{E} \left(\sum_k^K \sum_j^J \sum_i^I \left| \sum_{l=1}^\infty 1_{A_{ji}^k} 1_{\{\eta(B_j \times (s_{k-1}, s_k]) = l\}} l \xi_{ji}^k \right|^p + \sum_k^K \sum_j^J \sum_i^I \left| 1_{A_{ji}^k} \xi_{ji}^k \nu(B_j)\tau \right|^p \right)^{\frac{r}{p}} \\
\le\ & C_p(E)\, 2^r \\
&\mathbb{E} \left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=1}^\infty 1_{A_{ji}^k} 1_{\{\eta(B_j \times (s_{k-1}, s_k]) = l\}} l^p \left| \xi_{ji}^k \right|^p + \sum_k^K \sum_j^J \sum_i^I 1_{A_{ji}^k} \left| \xi_{ji}^k \nu(B_j)\tau \right|^p \right)^{\frac{r}{p}}.
\end{aligned}$$

Using $|x-y|^q \le 2^q\left(|x|^q+|y|^q\right)$ for $q=\frac{r}{p}$ we get

$$\begin{aligned}
\ldots \quad &\le \quad C_p(E)\, 2^{r+\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=1}^\infty 1_{A^k_{ji}} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, l^p\, \left|\xi^k_{ji}\right|^p\right)^{\frac{r}{p}} \\
&\qquad + C_p(E)\, 2^{r+\frac{r}{p}}\, \tilde{\mathbb{E}}\left(\sum_k^K \sum_j^J \sum_i^I \left|1_{A^k_{ji}} \xi^k_{ji} \nu(B_j)\tau\right|^p\right)^{\frac{r}{p}}.
\end{aligned}$$

Let m_0 be so large that $p-1\le m_0\frac{p}{r}$. We split again the inner term in the inner sum. Doing this we get

$$\begin{aligned}
\ldots \quad &\le C_p(E)\, 2^{r+\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=1}^{m_0-1} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, 1_{A^k_{ji}}\, l^p\, \left|\xi^k_{ji}\right|^p\right)^{\frac{r}{p}} \\
&\qquad + C_p(E)\, 2^{r+\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=m_0}^\infty 1_{A^k_{ji}} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}} l^p \left|\xi^k_{ji}\right|^p\right)^{\frac{r}{p}} \\
&\qquad + C_p(E)\, 2^{r+\frac{r}{p}}\, \tilde{\mathbb{E}}\left(\sum_k^K \sum_j^J \sum_i^I \left|1_{A^k_{ji}} \xi^k_{ji} \nu(B_j)\tau\right|^p\right)^{\frac{r}{p}}.
\end{aligned} \tag{3.4}$$

The first term in (3.4) can be estimated in the following way. Since $l\le(m_0-1)$, we put $(m_0-1)^{(p-1)\frac{r}{p}}$ in front of the braket. Next, we add the additional terms for $l=m_0,m_0+1,\ldots$ and then change again the representation

$$\begin{aligned}
\ldots \quad &\le \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I 1_{A^k_{ji}} \sum_{l=1}^{m_0-1} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, l^p \left|\xi^k_{ji}\right|^p\right)^{\frac{r}{p}} \\
&= \quad (m_0-1)^{(p-1)\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=1}^{m_0-1} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, 1_{A^k_{ji}}\, l\ \left|\xi^j_n\right|^p\right)^{\frac{r}{p}} \\
&\le \quad (m_0-1)^{(p-1)\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I \sum_{l=0}^{\infty} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, 1_{A^k_{ji}}\, l\ \left|\xi^j_n\right|^p\right)^{\frac{r}{p}} \\
&= \quad (m_0-1)^{(p-1)\frac{r}{p}}\, \mathbb{E}\left(\sum_k^K \sum_j^J \sum_i^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \eta(B_j\times(s_{k-1},s_k])\right)^{\frac{r}{p}} \\
&= \quad (m_0-1)^{(p-1)\frac{r}{p}}\, \mathbb{E}\left(\int_0^T \int_{S^\epsilon} |\xi(s,z)|^p \eta(dz,ds)\right)^{\frac{r}{p}} \\
&\le \quad (m_0-1)^{(p-1)\frac{r}{p}}\, \mathbb{E}\left(\int_0 \prod_k^K \prod_j^J \frac{\nu(B_j)^{l_{kj}}\tau^{l_{kj}}}{l_{kj}!}^T \int_S |\xi(s,z)|^p \eta(dz,ds)\right)^{\frac{r}{p}}.
\end{aligned}$$

Now we consider the second term of (3.4). First, the sets $B_j\times(s_{k-1},s_k]$ are disjoint, therefore the random variables $\eta(B_j\times(s_{k-1},s_k])$ independent. Secondly, the $\mathbb{N}_0$ valued random variables $\{\eta(B_j\times(s_{k-1},s_k])\}$ are Poisson distributed with parameter $\nu(B_j)\tau$, therefore, the explicit formula of the expectation gives

$$
\begin{aligned}
&\mathbb{E}\left(\sum_k^K\sum_j^J\sum_i^I\sum_{l=m_0}^\infty 1_{A^k_{ji}}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,l^p\left|\xi^k_{ji}\right|^p\right)^{\frac{r}{p}} = \\
&\quad\sum_{\mathfrak{l}\in\tilde\Omega}\mathbb{P}\left(\eta(B_j\times(s_{k-1},s_k])=l_{k,j},0\le j\le J,0\le k\le K\right)\\
&\quad\mathbb{E}\Big[\Big(\sum_k^K\sum_j^J\sum_i^I\sum_{l=m_0}^\infty 1_{A^k_{ji}}1_{\{l=\mathfrak{l}_{k,j}\}}\,\mathfrak{l}^p_{k,j}\left|\xi^k_{ji}\right|^p\Big)^{\frac{r}{p}}\\
&\qquad\mid\eta(B_j\times(s_{k-1},s_k])=\mathfrak{l}_{k,j},0\le j\le J,0\le k\le K\Big]\\
&=\sum_{\mathfrak{l}\in\tilde\Omega}\prod_k^K\prod_j^J\frac{\nu(B_j)^{l_{kj}}\tau^{l_{kj}}}{l_{kj}!}\,\mathbb{E}\Big[\Big(\sum_k^K\sum_j^J\sum_i^I\sum_{l=m_0}^\infty 1_{A^k_{ji}}1_{\{l=\mathfrak{l}_{k,j}\}}\,\mathfrak{l}^p_{k,j}\left|\xi^k_{ji}\right|^p\Big)^{\frac{r}{p}}\\
&\qquad\mid\eta(B_j\times(s_{k-1},s_k])=\mathfrak{l}_{k,j},0\le j\le J,0\le k\le K\Big]
\end{aligned}
$$

where we put $\tilde\Omega=\otimes_{k=1}^K\otimes_{j=1}^J\mathbb{N}_0$. Note that, since the sum over l starts at m_0, if $l_{kj}<m_0$, $1_{l==l_{k,j}\}}=0$. Therefore, for any $\mathfrak{l}$ which contributes to the sum we can put at least m_0 factors of the product in the front. Therefore, let $\mathfrak{e}(\mathfrak{l})_{jk}:=l_{kj}-m_0$ if $l_{jk}\ge m_0$ and $\mathfrak{e}(\mathfrak{l})_{jk}:=l_{kj}$ otherwise, and $\#\mathfrak{l}:=\{\mathfrak{l}_{k,j}\ge m_0:0\le k\le K,0\le j\le J\}$. Now, putting the factors of the product in front of the summands gives

$$
\begin{aligned}
&\le\max_{1\le j\le J}\nu(B_j)^{m_0}\tau^{m_0}\sum_{\mathfrak{l}\in\tilde\Omega}\prod_k^K\prod_j^J\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{l}_{ki}!}\\
&\quad\mathbb{E}\Big[\Big(\sum_k^K\sum_j^J\sum_i^I\sum_{l=m_0}^\infty 1_{A^k_{ji}}1_{\{l=\mathfrak{l}_{k,j}\}}\,\mathfrak{l}^p_{k,j}\left|\xi^k_{ji}\right|^p\Big)^{\frac{r}{p}}\\
&\qquad\mid\eta(B_j\times(s_{k-1},s_k])=\mathfrak{l}_{k,j},0\le j\le J,0\le k\le K\Big]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \max_{1\le j\le J} \nu(B_j)^{m_0}\tau^{m_0} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_k^K \prod_j^J \frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
& \mathbb{E}\Bigg[\Big(\sum_k^K \sum_j^J \sum_i^I \sum_{l=m_0}^\infty 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}} \frac{\mathfrak{l}_{k,j}^p}{\mathfrak{l}_{k,j}^{\frac{p}{r}}(\mathfrak{l}_{kj}-1)^{\frac{p}{r}}\cdots(\mathfrak{l}_{kj}-m_0)^{\frac{p}{r}}} \left|\xi_{ji}^k\right|^p\Big)^{\frac{r}{p}} \\
& \qquad \mid \eta(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\Bigg].
\end{aligned}
$$

If $p - m_0\frac{p}{r} \le 1$ there exists a constant $C>0$ such that

$$
\frac{l^p}{l^{\frac{p}{r}}(l-1)^{\frac{p}{r}}\cdots(l-m_0)^{\frac{p}{r}}} \le C\,(l-m_0), \quad l\ge m_0.
$$

Hence,

$$
\begin{aligned}
\cdots \quad \le \quad & C \max_{1\le j\le J} \nu(B_j)^{m_0}\tau^{m_0} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_k^K \prod_j^J \frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
& \mathbb{E}\Bigg[\Big(\sum_k^K \sum_j^J \sum_i^I \sum_{l=m_0}^\infty 1_{A_{ji}^k} 1_{\{l=l_{k,j}\}}\,(l-m_0)\left|\xi_{ji}^k\right|^p\Big)^{\frac{r}{p}} \\
& \qquad \mid \eta(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\Bigg].
\end{aligned}
$$

Renumerating gives

$$
\begin{aligned}
\cdots \quad \le \quad & C \max_{1\le j\le J} \nu(B_j)^{m_0}\tau^{m_0} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_k^K \prod_j^J \frac{\nu(B_j)^{\mathfrak{l}_{kj}}\tau^{\mathfrak{l}_{ki}}}{\mathfrak{l}_{ki}!} \\
& \mathbb{E}\Bigg[\Big(\sum_k^K \sum_j^J \sum_i^I \sum_{l=0}^\infty 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l\left|\xi_{ji}^k\right|^p\Big)^{\frac{r}{p}} \\
& \qquad \mid \eta(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\Bigg].
\end{aligned}
$$

Going back gives

$$
\begin{aligned}
\cdots \quad \le \quad & C \max_{1\le j\le J} \nu(B_j)^{m_0}\tau^{m_0} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_k^K \prod_j^J \mathbb{P}\left(\eta(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\right) \\
& \mathbb{E}\Bigg[\Big(\sum_k^K \sum_j^J \sum_i^I \sum_{l=0}^\infty 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l\left|\xi_{ji}^k\right|^p\Big)^{\frac{r}{p}} \\
& \qquad \mid \eta(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad C \max_{1\leq j\leq J} \nu(B_j)^{m_0} \tau^{m_0} \sum_{\mathfrak{l}\in\tilde\Omega} \prod_k^K \prod_j^J \frac{\nu(B_j)^{\mathfrak{l}_{kj}} \tau^{\mathfrak{l}_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
&\qquad \mathbb{E}\Bigg[\Big(\sum_k^K \sum_j^J \sum_i^I \sum_{l=0}^\infty 1_{A^k_{ji}} 1_{\{l=l_{k,j}\}}\, l \left|\xi^k_{ji}\right|^p \Big)^{\frac{r}{p}} \\
&\qquad\qquad \mid \eta(B_j\times(s_{k-1},s_k]) = l_{k,j}, 0\leq j\leq J, 0\leq k\leq K \Bigg] \\
&= \quad C \max_{1\leq j\leq J} \nu(B_j)^{m_0}\, \tau^{m_0}\, \mathbb{E}\left(\int_0^t \int_{S^\epsilon} |\xi(z,s)|\, \eta(dz,ds)\right)^{\frac{r}{p}}.
\end{aligned}
$$

Using the assumption $\nu(B_j)\leq 1$ for all $j=1,\ldots,J$, we obtain

$$
= \quad C\ \tau^{m_0}\, \mathbb{E}\left(\int_0^t \int_S |\xi(z,s)|\, \eta(dz,ds)\right)^{\frac{r}{p}}.
$$

It remains to investigate the last summand in (3.4). Observe that, first, since $\nu(B_j)\leq 1$, $\nu(B_j)^p \leq \nu(B_j)$, and, secondly, $\sum_j^J \nu(B_j) = \nu(S_\epsilon)$. Thus, applying the Hölder inequality twice, and then, again, taken into account that $\{A^k_{ji}, 1\leq i\leq I\}$ are disjoint, we put the sum running over i in front of the brackets. Doing so, we arrive at

(3.5)

$$
\begin{aligned}
\tilde{\mathbb{E}} \left(\sum_k^K \sum_j^J \sum_i^I \left| 1_{A^k_{ji}} \xi^k_{ji} \nu(B_j)\tau \right|^p \right)^{\frac{r}{p}} &\leq \tilde{\mathbb{E}} \left(\sum_k^K \sum_j^J \sum_i^I \tau^p 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \nu(B_j)^p \right)^{\frac{r}{p}} \\
\leq \quad \tau^{(p-1)\frac{r}{p}} \max_j \nu(B_j)^{(p-1)\frac{r}{p}}\ \nu(S_\epsilon)^{\frac{p}{r}}\, \tilde{\mathbb{E}} \sum_k^K \tau & \left(\sum_j^J \sum_i^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \frac{\nu(B_j)}{\nu(S_\epsilon)} \right)^{\frac{r}{p}} \\
\leq \quad \tau^{(p-1)\frac{r}{p}} \max_j \nu(B_j)^{(p-1)\frac{r}{p}} \nu(S_\epsilon)^{\frac{p}{r}}\, \tilde{\mathbb{E}} \sum_k^K \tau \sum_j^J & \left(\sum_i^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \right)^{\frac{r}{p}} \frac{\nu(B_j)}{\nu(S_\epsilon)} \\
\leq \quad \tau^{r-\frac{r}{p}} \max_j \nu(B_j)^{(p-1)\frac{r}{p}} \nu(S_\epsilon)^{\frac{p}{r}} \tilde{\mathbb{E}} \sum_k^K \tau \sum_j^J \sum_i^I & 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^r \frac{\nu(B_j)}{\nu(S^\epsilon)}. \\
\leq \quad \tau^{r-\frac{r}{p}} \max_j \nu(B_j)^{(p-1)\frac{r}{p}} \nu(S_\epsilon)^{\frac{p}{r}-1} \tilde{\mathbb{E}} \sum_k^K \tau \sum_j^J \sum_i^I & 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^r \nu(B_j).
\end{aligned}
$$

The RHS of (3.5) is bounded by

$$
\nu(S_\epsilon)^{\frac{p}{r}-1}\, \tau^{r-\frac{r}{p}} \int_0^T \int_{S_\epsilon} \mathbb{E}\, |\xi(s,z)|^r\, \nu(dz)\, ds.
$$

Collecting all together we arrive at

$$\mathbb{E}\left|\int_0^T\int_{S^\epsilon}\xi(s,x)\tilde{\eta}(dx,ds)\right|^r \leq$$
$$C_p(E)2^{r+\frac{r}{p}}(1+\tau)\mathbb{E}\left(\int_0^T\int_S|\xi(s,x)|^p\,\eta(dx,ds)\right)^{\frac{r}{p}}$$
$$+\,C_p(E)2^{r+\frac{r}{p}}\nu(S^\epsilon)^{\frac{p}{r}-1}\tau^{p-1}\int_0^T\int_S\mathbb{E}\,|\xi(s,z)|^r\,\nu(dz)\,ds.$$

It remains to take the limit. But, since $r>p$, $\nu(S_\epsilon)^{\frac{p}{r}-1}\to 0$ as $\epsilon\to 0$, we obtain

$$\mathbb{E}\left|\int_0^T\int_S\xi(s,x)\tilde{\eta}(dx,ds)\right|^r \leq \tag{3.6}$$
$$C_p(E)\,2^{r+\frac{r}{p}}(1+\tau)\,\mathbb{E}\left(\int_0^T\int_S|\xi(s,x)|^p\,\eta(dx,ds)\right)^{\frac{r}{p}}$$

In the second step we assume that $\xi\in\mathcal{M}^r(\mathbb{R}_+;L^r(S,\nu;E))$ is approximated by a sequence of simple functions $(\xi_n)_{n\in\mathbb{N}}$, where we take in time the shifted Haar projection of order n and in space an arbitrary simple function. Therefore, let ξ_n, $n\in\mathbb{N}$, be a sequence of simple functions, such that ξ_n is constants on the dyadic intervals $[2^{-n}k,2^{-n}(k+1))$ and $\xi_n\to\xi$ in $\mathcal{M}^r(\mathbb{R}_+;L^r(S,\nu;E))$. Substituting ξ_n in (3.6) we obtain

$$\mathbb{E}\left|\int_0^T\int_{S_\epsilon}\xi(s,x)\tilde{\eta}(dx,ds)\right|^r \leq C_p(E)\,2^{r+\frac{r}{p}}(1+2^{-n})\times$$
$$\left\{\mathbb{E}\left(\int_0^T\int_S|\xi(s,x)|^p\,\eta(dx,ds)\right)^{\frac{r}{p}}+\nu(S_\epsilon)^{\frac{p}{r}-1}2^{-n(p-1)}\int_0^T\int_S\mathbb{E}\,|\xi(s,z)|^r\,\nu(dz)\,ds\right\}.$$

Taking the limit for n to infinity we get

$$\mathbb{E}\left|\int_0^T\int_{S_\epsilon}\xi(s,x)\tilde{\eta}(dx,ds)\right|^r \leq C_p(E)\,2^{r+\frac{r}{p}}\,\mathbb{E}\left(\int_0^T\int_S|\xi(s,z)|^p\,\eta(dz,ds)\right)^{\frac{r}{p}}. \tag{3.7}$$

In the third and last step we let ϵ converge to zero. Since the RHS of (3.7) is independent of ϵ, the assertion is shown. □

Proof of Inequality (iii): The proof is a generalisation of the proofs of Bass and Cranston [4, Lemma 5.2] or Protter and Talay [19, Lemma 4.1]. It follows from Inequality (ii) that

$$\mathbb{E}\sup_{0<s\leq t}\left|\int_0^t\int_S\xi(s;z)\tilde{\eta}(dz;ds)\right|^{p^n}$$
$$\leq\quad C_p(E)\,2^{p^n+p^{n-1}}(m_0-1)^{(p-1)\frac{r}{p}}\,\mathbb{E}\left(\int_0^t\int_S|\xi(s;z)|^p\;\eta(dz;ds)\right)^{p^{n-1}}.$$

Simple calculations lead to

$$\begin{aligned}\mathbb{E}\sup_{0<s\le t}\left|\int_0^t\int_S \xi(s;z)\tilde\eta(dz;ds)\right|^{p^n} &\le C_p(E)\,2^{p^n+2p^{n-1}}\,(m_0-1)^{(p-1)\frac{r}{p}}\times\\ \le\quad &\left(\mathbb{E}\left(\int_0^t\int_S|\xi(s,z)|^p\;\tilde\eta(dz;ds)\right)^{p^{n-1}}+\mathbb{E}\left(\int_0^t\int_S|\xi(s,z)|^p\;\gamma(dz;ds)\right)^{p^{n-1}}\right)\end{aligned}$$

Let us define

$$L(t)^{(0)}:=\int_0^t\int_S|\xi(s,z)|^p\;\tilde\eta(dz;ds),\quad t\ge 0.$$

Then,

$$\begin{aligned}\mathbb{E}\sup_{0<s\le t}\left|\int_0^t\int_S \xi(s;z)\tilde\eta(dz;ds)\right|^{p^n} &\le C_p(E)\,2^{p^n+2p^{n-1}}\,(m_0-1)^{(p-1)\frac{r}{p}}\times\\ &\left(\mathbb{E}|L(t)^{(0)}|^{p^{n-1}}+\mathbb{E}\left(\int_0^t\int_S|\xi(s,z)|^p\;\nu(dz)\,ds\right)^{p^{n-1}}\right).\end{aligned}\tag{3.8}$$

If n equals 2 we are done. In particularly, Inequality (i) for $r=p$ gives

$$\begin{aligned}\mathbb{E}\left|L(t)^{(0)}\right|^p &\le \mathbb{E}\left|\int_0^t\int_S|\xi(s,z)|^p\;\tilde\eta(dz;ds)\right|^p\\ &\le 2^{2-p}\,\mathbb{E}\int_0^t\int_S|\xi(s,z)|^{p^2}\;\nu(dz)\,ds.\end{aligned}\tag{3.9}$$

Substituting (3.9) in (3.8) we get for $n=2$

$$\begin{aligned}\mathbb{E}\sup_{0<s\le t}\left|\int_0^t\int_S \xi(s;z)\tilde\eta(dz;ds)\right|^{p^2} &\le C_p(E)\,2^{p^2+2p}\times\\ &\left(2^{2-p}\,\mathbb{E}\int_0^t\int_S|\xi(s,z)|^{p^2}\;\nu(dz)\;ds+\mathbb{E}\left(\int_0^t\int_S|\xi(s,z)|^p\;\nu(dz)\;ds\right)^p\right).\end{aligned}$$

Now, Inequality (iii) is proved, provided $n=2$. If $n>2$, then we have to continue. Let

$$L(t)^{(r)}:=\int_0^t\int_{\mathbb{R}}|\xi(s,z)|^{p^{r+1}}\;\tilde\eta(dz;ds)\qquad\text{for } r=1,\ldots,n.$$

Inequality (ii) leads to

$$\begin{aligned}\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} &=\mathbb{E}\left(\int_0^t\int_{\mathbb{R}}|\xi(s,z)|^{p^{r+1}}\tilde\eta\,(dz;ds)\right)^{p^m}\\ \le\quad & C_p(E)\,2^{p^m+p^{m-1}}\,(m_0-1)^{(p-1)\frac{r}{p}}\;\mathbb{E}\left(\int_0^t\int_{\mathbb{R}}|\xi(s,z)|^{p^{r+2}}\eta(dz;ds)\right)^{p^{m-1}}.\end{aligned}$$

Since $\gamma = \nu \times \lambda$, simple calculations lead to

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} &\leq C_p(E)\, 2^{p^m+p^{m-1}} \\
&\quad \mathbb{E}\left(\int_0^t\int_{\mathbb{R}} |\xi(s,z)|^{p^{r+2}}\tilde{\eta}(dz;ds) + \int_0^t\int_S |\xi(s,z)|^{p^{r+2}}\gamma(dz;ds)\right)^{p^{m-1}} \\
&\leq C_p(E)\, 2^{p^m+2p^{m-1}}(m_0-1)^{(p-1)\frac{r}{p}} \\
&\quad \left(\mathbb{E}\left|L(t)^{(r+1)}\right|^{p^{m-1}} + \mathbb{E}\left(\int_0^t\int_S |\xi(s;z)|^{p^{r+2}}\ \nu(dz)\,ds\right)^{p^{m-1}}\right). \qquad (3.10)
\end{aligned}$$

Starting with $L(t)^{(0)}$ and iterating the calculation done in (3.10) lead for arbitrary n to

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(0)}\right|^{p^n} &\leq C_p(E)\, 2^{p^n+2p^{n-1}}(m_0-1)^{(p-1)\frac{r}{p}} \\
&\quad \left(\mathbb{E}\left|L(t)^{(1)}\right|^{p^{n-1}} + \mathbb{E}\left(\int_0^t\int_S |\xi(s;z)|^{p}\ \nu(dz)\,ds\right)^{p^{n-1}}\right) \\
&\leq C_p(E)\, 2^{p^n+2p^{n-1}}\, 2^{p^{n-1}+2p^{n-2}}(m_0-1)^{(p-1)\frac{r}{p}}\,\mathbb{E}\left|L(t)^{(2)}\right|^{p^{n-2}} \\
&\quad + \sum_{i=1}^{2}\bar{C}(i)\,\mathbb{E}\left(\int_0^t\int_S |\xi(s;z)|^{p^i}\ \nu(dz)ds\right)^{p^{n-i}} \\
&\leq C_p(E) 2^{p^n+2p^{n-1}}(m_0-1)^{(p-1)\frac{r}{p}}\,\mathbb{E}\left|L(t)^{(n-1)}\right|^{p} + \qquad (3.11) \\
&\quad \sum_{i=1}^{n-1}\bar{C}(i)\,\mathbb{E}\left(\int_0^t\int_S |\xi(s,z)|^{p^i}\ \nu(z)ds\right)^{p^{n-i}}, \qquad (3.12)
\end{aligned}$$

where $\bar{C}(0)=1$ and $\bar{C}(i) = \bar{C}(i-1)2^{p^{n-i+1}+2p^{n-1}}(m_0-1)^{(p-1)\frac{r}{p}}\, m(n-i)$. Finally, Inequality (i) gives

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(n-1)}\right|^{p} &= \mathbb{E}\left|\int_0^t\int_S |\xi(s,z)|^{p^n}\ \tilde{\eta}(dz;ds)\right|^{p} \\
&\leq 2^{2-p}\,\mathbb{E}\int_0^t\int_S |\xi(s,z)|^{p^{n+1}}\nu(dz)\,ds. \qquad (3.13)
\end{aligned}$$

Thus, substituting (3.13) in (3.11) we arrive at

$$\begin{aligned}
\mathbb{E}\sup_{0<s\leq t}\left|\int_0^t\int_S \xi(s;z)\tilde{\eta}(dz;ds)\right|^{p^n} &\leq C_p(E)2^{p^n+2p^{n-1}}(m_0-1)^{(p-1)\frac{r}{p}} \\
&\quad \left(\mathbb{E}|L(t)^{(0)}|^{p^{n-1}} + \mathbb{E}\left(\int_0^t\int_S |\xi(s,z)|^{p}\,\nu(dz)\,ds\right)^{p^{n-1}}\right) \\
&\leq 2^{2-p}\sum_{i=1}^{n}\bar{C}(i)\,\mathbb{E}\left(\int_0^t\int_S |\xi(s,z)|^{p^i}\ \nu(z)ds\right)^{n-i}.
\end{aligned}$$

$\square$

Appendix A. Discrete Inequalities of Burkholder-Davis-Gundy type

In this section we collect some basic informations about the martingale type p, $p\in[1,2]$, Banach spaces. For more details we refer to [6, Appendix C] or [5]. A property which encompasses both, the UMD property and the type p property is the martingale type p property.

Definition A.1. *Assume that $p\in[1,2]$ is fixed. A Banach space E is of martingale type p iff there exists a constant $L_p(E)>0$ such that for all E-valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds*

$$\sup_{0\le n\le N}\mathbb{E}|M_n|_E^p \le L_p(E)\,\mathbb{E}\sum_{n=0}^N |M_n-M_{n-1}|_E^p,$$

where as usually, we put $M_{-1}=0$.

A useful tool in the theory of martingales is the Doob's maximal inequality. The simplest version says that all real valued non-negative submartingales $\{M_n\}_{n=0}^N$ satisfy the inequality,

$$\lambda\mathbb{P}\left(\sup_{0\le k\le n}|M_k|>\lambda\right)\le \mathbb{E}\,1_{\max_{k\le n}M_k\ge\lambda}\,|M_n|,\quad 1\le n\le N,$$

and, hence, satisfy

$$\lambda^p\,\mathbb{P}\left(\sup_{0\le k\le n}|M_k|^p>\lambda\right)\le \mathbb{E}|M_n|^p,\quad 1\le n\le N. \tag{A.1}$$

Now, one gets immediately that all real valued non-negative submartingales $\{M_n\}_{n=0}^N$ satisfy

$$\mathbb{E}|M_n|^p\le \mathbb{E}\sup_{0\le k\le n}|M_k|^p\le q^p\,\mathbb{E}|M_n|^p, \tag{A.2}$$

where q is the conjugate exponent to p. From the last version of Doob's maximal inequality we can derive the following corollary.

Corollary A.2. *Let $p\in[1,2]$ and let E be a Banach space of martingale type p. Then there exist a constant $C=C_p(E)$ such that for all E-valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds*

$$\mathbb{E}\sup_{0\le n\le N}|M_n|_E^p \quad\le\quad C\,\mathbb{E}\sum_{k=0}^N|M_n-M_{n-1}|_E^p, \tag{A.3}$$

where as usually, we put $M_{-1}=0$.

Nevertheless, in the proof of inequality (ii) we used a stronger inequality, namely, we supposed that there exists a constant C such that for all E-valued finite martingales $\{M_n\}_{n=0}^N$ the following inequality holds

$$\mathbb{E}|\sum_n M_n|^r \quad\le\quad C\mathbb{E}\left(\sum_{n=0}^N|M_n-M_{n-1}|^p\right)^{\frac{r}{p}}. \tag{A.4}$$

This stronger inequality we can derive from a generalisation of Doob's maximal inequality. But before showing inequality (A.4), since it is interesting on its own, we state the generalization of Doob's maximal inequality. To be more precise, in the Doob's maximal inequality we can

replace the square by a convex, non decreasing and continuous function $\Phi : [0,\infty) \to \mathbb{R}$ with $\Phi(0) = 0$. In addition, Φ has to satisfy *the growth condition* (see Appendix)

Proposition A.3. *(Garsia* [11, p.173]*) For all convex, non decreasing, and continuous function $\Phi : [0,\infty) \to \mathbb{R}$ with $\Phi(0) = 0$ and satisfying the growth condition, there exists a constant C such that for all non-negative real valued sub-martingales $\{X_n\}_{n=0}^N$ with $X_0 = 0$ we have*

$$\mathbb{E}\Phi\Big(\sup_{1\le n\le N} X_n\Big) \le C\,\mathbb{E}\Phi\big(X_N\big).$$

To be precise, $C = 4\,(C_\Psi^* - 1)$ where Ψ is the conjugate convex function to Φ.

But before starting with the proof we state a result of Garsia, i.e. [11, Theorem 2.1].

Theorem A.4. *If $\{X_n\}_{n=0}^N$ is a nonnegative real valued sub-martingale with $X_0 = 0$ and if $\phi : [0,\infty) \to [0,\infty)$ is a non decreasing function, then*

$$\mathbb{E}\int_0^{X_n^*} t\,d\phi(t) = \mathbb{E}X_n\phi(X_n^*), \quad n \in \mathbb{N},$$

where $X_n^* = \sup_{k\le n}|X_k|$, $n \in \mathbb{N}$.

Now, we can start with the proof.

Proof of Proposition A.3. Now, since $\Psi(\phi(t)) = \int_0^t s\,d\psi(s)$, we obtain

$$\mathbb{E}\Psi(\phi(X_n^*)) = \mathbb{E}\int_0^{X_n^*} s\,d\psi(s).$$

Theorem A.4 gives

$$\mathbb{E}\Psi(\phi(X_n^*)) = \mathbb{E}X_n\phi(X_n^*).$$

The Young inequality gives

$$\mathbb{E}\Psi(\phi(X_n^*)) = 2\,\mathbb{E}\Phi(X_n) + \frac{1}{2}\,\mathbb{E}\Psi(\phi(X_n^*)). \tag{A.5}$$

Subtracting $\frac{1}{2}\,\mathbb{E}\Psi(\phi(X_n^*)$ on both sides of inequality (A.5) gives

$$\frac{1}{2}\,\mathbb{E}\Psi(\phi(X_n^*)) \le 2\,\mathbb{E}\Phi(X_n).$$

By property B.8 we get the assertion. □

Assume E is of martingale type p. Now, from the Definition A.1 and the generalized Doob maximal inequality, i.e. Proposition A.3, we can show that the Burkholder-Davis-Gundy inequality is also valid on E.

Theorem A.5. *Let $\Phi : [0,\infty) \to \mathbb{R}$ be a non decreasing, convex and continuous function with $\Phi(0) = 0$ and satisfying the growth condition (for definition we refer to B).*
Let $p \in [1,2]$ be fixed and let E be a Banach space E of martingale type p. Then, there exists a constant $C_p(E,\phi) > 0$ such that for all E-valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\mathbb{E}\Phi\left(\sup_{0\le n\le N}|M_n|_E\right) \le C_p(E,\phi)\,\mathbb{E}\Phi\left(\left(\sum_{n=0}^N |M_n - M_{n-1}|_E^p\right)^{\frac{1}{p}}\right),$$

where as usually, we put $M_{-1} = 0$.

We will need the following Lemmata. Since the Lemmata are valid for real valued random variables, we omit their proofs and give only the reference.

Lemma A.6. *(Burkholder, Davis and Gundy* [8, Theorem 3.2]*, Garsia* [11, Theorem 0.1]*) Let Φ be a convex function satisfying the conditions of Theorem A.5 and $(\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n=0}^N,\mathbb{P})$ be a filtered probability space. Then there exists a constant C, only depending on Φ, such that for all sequence $\{z_n\}_{n=0}^N$ of real-valued, non negative and measurable functions $(\Omega,\mathcal{F},\mathbb{P})$ the following inequality holds*

$$\mathbb{E}\,\Phi\left(\sum_{n=0}^N \mathbb{E}[z_n\mid\mathcal{F}_{n-1}]\right)\le C\,\mathbb{E}\Phi\left(\sum_{n=0}^N z_n\right).$$

To be more precise, $C=(c_\Phi^*)^{2c_\Phi^*}$, where c^*_Φ is defined in (B.5)*.*

Proof of Lemma A.6. We are following the proof of Garsia [11, Theorem 0.1]. Put for ϕ given by ...

$$Z_n:=\sum_{k=0}^n z_k,\qquad Z_n^{\mathcal{F}}:=\sum_{k=0}^n \mathbb{E}[z_k\mid\mathcal{F}_k]$$

$$Y_0=0,\qquad Y_k:=\mathbb{E}[\phi(Z_n^{\mathcal{F}})\mid\mathcal{F}_k],\ 0\le k\le n.$$

This given we obtain by the tower property

$$\mathbb{E}Z_n^{\mathcal{F}}\phi(Z_n^{\mathcal{F}})=\mathbb{E}\sum_{k=0}^n\mathbb{E}[z_k\mid\mathcal{F}_k]\cdot\phi(Z_n^{\mathcal{F}})=\mathbb{E}\sum_{k=0}^n\mathbb{E}[\,\mathbb{E}[z_k\mid\mathcal{F}_k]\cdot\phi(Z_n^{\mathcal{F}})\mid\mathcal{F}_k]$$

$$=\mathbb{E}\sum_{k=0}^n\mathbb{E}[z_k\mid\mathcal{F}_k]\cdot\mathbb{E}[\phi(Z_n^{\mathcal{F}})\mid\mathcal{F}_k]=\mathbb{E}\sum_{k=0}^n z_k\,\mathbb{E}[\phi(Z_n^{\mathcal{F}})\mid\mathcal{F}_k]\le\mathbb{E}\sum_{k=0}^n z_k\,Y_n^*\le Z_nY_n^*.$$

From the Young inequality we get for $a=c_\phi^*$ (for definition of c_ϕ^* see (B.5))

$$\mathbb{E}Z_n^{\mathcal{F}}\phi(Z_n^{\mathcal{F}})\le\mathbb{E}\Phi(a^2Z_n)+\mathbb{E}\Psi(a^{-2}Y_n^*)$$

From properties of Φ, i.e. (B.1) we get

$$\mathbb{E}\Phi(Z_n^{\mathcal{F}})+\mathbb{E}\Psi(\phi(Z_n^{\mathcal{F}}))\le\mathbb{E}\Phi(a^2Z_n)+a^{-2}\,\mathbb{E}\Psi(Y_n^*)\le\mathbb{E}\Phi(a^2Z_n)+a^{-1}\,\mathbb{E}\Psi(aY_n).$$

From and the definition of Y, we get

$$\mathbb{E}\Phi(Z_n^{\mathcal{F}})+\mathbb{E}\Psi(\phi(Z_n^{\mathcal{F}}))\le(c_\Phi^*)^{2c_\phi^*}\mathbb{E}\Phi(Z_n)+\mathbb{E}\Psi(Y_n)\le(c_\Phi^*)^{2c_\phi^*}\mathbb{E}\Phi(Z_n)+\mathbb{E}\Psi(\phi(Z_n^{\mathcal{F}})).$$

Subtraction on both sides $\mathbb{E}\Psi(\phi(Z_n^{\mathcal{F}}))$ leads the assertion.

□

Lemma A.7. [7, Lemma 7.1] *Suppose that x and y are nonnegative $\mathbb{R}$-valued random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $\beta>1$, $\delta>1$, $\epsilon>1$ are real numbers such that*

$$\mathbb{P}\left(y>\beta\lambda,\,x\le\delta\lambda\right)\le\epsilon\,\mathbb{P}\left(y>\lambda\right),\quad\lambda>0.$$

In addition, let γ and η be real numbers satisfying

$$\Phi(\beta\lambda)\le\gamma\Phi(\lambda),\quad\Phi(\lambda/\delta)\le\eta\Phi(\lambda),\quad\lambda\ge0.$$

If $\gamma\epsilon<1$ then

$$\mathbb{E}\Phi(y)\le\frac{\gamma\eta}{1-\gamma\epsilon}\mathbb{E}\Phi(x).$$

Proof of Lemma A.7. The proof follows by some direct calculations, therefore, we omit the proof and refer the reader e.g. to [7, Lemma 7.1]. □

By means of the generalised Doob's maximal inequality and the Lemmata before, following, for the Proof of Theorem A.5 necessary, Proposition can be verified.

Proposition A.8. *There exists a constant $C < \infty$ such that for all E-valued martingales $\{M_n\}_{n=0}^N$ and all M-previsible processes $\{w_m\}_{m=0}^N$ satisfying $|M_n - M_{n-1}| \le w_n$ for all $1 \le n \le N$, we have*

$$\mathbb{E}\Phi(M_n^*) \le C\mathbb{E}\phi(S_{n,p}(M)) + C\mathbb{E}\phi(w_n^*), \quad 1 \le n \le N.$$

An estimate of the constant is given by $C_{\Phi,p}(E) = \min\{2\delta^{-c_\Phi^*}\beta^{c_\Phi^*} : \beta > 1, 0 < \delta < 1 - \beta$ such that $2L_p(E)\frac{\delta^p\beta^{c_\Phi^*}}{(\beta-\delta-1)^p} = \frac{1}{2}\}$.

Proof of Proposition A.8. The proof follows the proof of [7, Theorem 15.1], where only the real valued case is considered. Therefore, we had to modify the original proof of Burkholder at some points. Without loss of generality we set $n = N$. Similarly, we will show that the random variables M_N^* and $S_{N,p}(M) \vee w_N^*$ satisfies the assumption of Lemma A.7. I.e. we will show that for $\beta > 1$ and $0 < \delta < \beta - 1$ the following holds

(A.6)
$$\mathbb{P}\left(M_N^* > \beta\lambda,\, S_{N,p}(M) \vee w_N^* \le \delta\lambda\right) \le 2\, L_p(X)\, \frac{\delta^p}{(\beta-\delta-1)^p}\, \mathbb{P}\left(M_N^* > \lambda\right), \quad \lambda > 0.$$

To prove (A.6), we introduce the following stopping times. Let $\mu := \inf\{1 \le n \le N : |M_n| > \lambda\}$, $\nu := \inf\{1 \le n \le N : |M_n| > \beta\lambda\}$, and $\sigma := \inf\{1 \le n \le N : |S_{n,p}(M)| > \delta\lambda$ or $w_n > \delta\lambda\}$. If the infimum will not be attained, we set the stopping time to ∞. Let $H = \{H_n,\, 1 \le n \le N\}$ be defined by

$$H_n := \sum_{k\le n} h_k = \sum_{k\le n} 1_{\{\mu<k\le\nu\vee\sigma\}}\,(M_k - M_{k-1}), \quad 0 \le n \le N.$$

Since w is previsible, $\{\mu < k \le \nu \vee \sigma\} \in \mathcal{F}_k^M$, and the process H is a martingale. Moreover, on $\{\mu = \infty\} = \{M_N^* \le \lambda\}$, $S_{N,p}(H) = 0$. On $\{0 < \sigma < \infty\}$, the assumption on w leads to

$$\begin{aligned} S_{N,p}^p(H) \le S_{\sigma,p}^p(H) \le S_{\sigma,p}^p(M) &= S_{\sigma-1,p}^p(M) + (M_\sigma - M_{\sigma-1})^p \\ &\le S_{\sigma-1,p}^p(M) + w_\sigma^p \le 2\delta^p\lambda^p. \end{aligned} \tag{A.7}$$

This inequality can also be extended to the case where $\sigma = \infty$. Therefore, since E is a Banach space of martingale type p, by Definition (A.1), there exists a constant $L_p(E)$ such that

$$\mathbb{E}|H_N|^p \le L_p(E)\,\mathbb{E}\sum_{k=0}^N |h_k|^p \le L_p(E)\,\mathbb{E}S_{N,p}^p(H). \tag{A.8}$$

Substituting (A.7), we get

$$\mathbb{E}|H_N|^p \le 2\,L_p(E)\,\delta^p\lambda^p\mathbb{P}\left(M_N^* > \lambda\right).$$

Since $\{M_N^* > \beta\lambda,\, S_{N,p}(M) \vee w_N^* \le \delta\lambda\} \subset \{H_N^* > \beta\lambda - \lambda - \delta\lambda\}$,

$$\mathbb{P}\left(M_N^* > \beta\lambda,\, S_{N,p}(M) \vee w_N^* \le \delta\lambda\right) \quad \le \quad \mathbb{P}\left(H_N^* > \beta\lambda - \lambda - \delta\lambda\right).$$

The simple version of Doob's maximal inequality, i.e. (A.1), and (A.8) give

$$\begin{aligned}\mathbb{P}\left(M_N^*>\beta\lambda,\, S_{N,p}(M)\vee w_N^*\le\delta\lambda\right) &\le \mathbb{P}\left(H_N^*>\beta\lambda-\lambda-\delta\lambda\right)\\ \le \frac{1}{(\beta-\delta-1)^p\lambda^p}\mathbb{E}|H_N|^p &\le 2\,L_p(E)\,\frac{\delta^p}{(\beta-\delta-1)^p}\mathbb{P}\left(M_N^*>\lambda\right).\end{aligned}$$

Since δ can be chosen arbitrary small, the assumptions of Lemma A.7 are satisfied and we apply the Lemma to verify the assertion. The exact constant can be verified by using inequality (B.7) and Definition (B.5). □

Proof of Theorem A.5. The proof works in analogy to the proof of Davis, Burkholder and Gundy (see e.g. [8, Theorem 1.1], or [7, Theorem 15.1]). For an E-valued martingale $\{M_n\}_{n=0}^N$ and its sequence of martingale differences $\{m_n\}_{n=0}^N$, let $\mathcal{A}_n:=\{|m_n|\le 2m_{n-1}^*\}$, $0\le n\le N$. Davis introduced in [10] the following decomposition of M, where $M=G+H$, and $G=\{G_n\}_{n=0}^N$ and $H=\{H_n\}_{n=0}^N$ are defined by

$$\begin{aligned}G_n &= \sum_{k=1}^n g_k:=\sum_{k=1}^n\left[y_k-\mathbb{E}[y_k\mid\mathcal{F}_{k-1}]\right],\quad 0\le n\le N,\\ H_n &= \sum_{k=1}^n h_k:=\sum_{k=1}^n\left[z_k+\mathbb{E}[y_k\mid\mathcal{F}_{k-1}]\right],\quad 0\le n\le N,\end{aligned}$$

with $y_k=m_k 1_{\mathcal{A}_k}$ and $z_k=m_k 1_{\mathcal{A}_k^C}$, $0\le k\le N$.
Now, since

$$M_n^*\le G_n^*+H_n^*,$$

by (B.6), there exists a constant $c_\Phi<\infty$, depending only on Φ, such that

$$\mathbb{E}\Phi(M_n^*)\le c_\Phi\,\mathbb{E}\Phi\big(G_n^*\big)+c_\Phi\,\mathbb{E}\Phi\big(H_n^*\big).\tag{A.9}$$

First, we will investigate the last term, i.e. $\mathbb{E}\Phi(H^*)$ and then we will investigate $\mathbb{E}\Phi(G^*)$. Observe that, since $\{m_n\}_{n=0}^N$ is a sequence of martingale differences,

$$\mathbb{E}\left[y_k\mid\mathcal{F}_{k-1}\right]+\mathbb{E}\left[z_k\mid\mathcal{F}_{k-1}\right]=\mathbb{E}\left[m_k\mid\mathcal{F}_{k-1}\right]=0,\quad 1\le k\le N.$$

Therefore, $H_n=\sum_{k=0}^n\left[z_k-\mathbb{E}[z_k\mid\mathcal{F}_{k-1}]\right]$, $1\le n\le N$. Applying the Jensen inequality and Lemma A.6 we get

$$\begin{aligned}\mathbb{E}\Phi\big(H_n^*\big) &\le \mathbb{E}\Phi\Big(\sum_{k=0}^n\left|z_k-\mathbb{E}[z_k\mid\mathcal{F}_{k-1}]\right|\Big)\\ &\le \mathbb{E}\Phi\Big(\sum_{k=0}^n|z_k|+\sum_{k=0}^n\mathbb{E}[|z_k|\mid\mathcal{F}_{k-1}]\Big)\le 2\,\mathbb{E}\Phi\Big(\sum_{k=0}^n|z_k|\Big).\end{aligned}$$

Since $|m_k|>2m_{k-1}^*$ implies $|m_k|<2(m_k^*-m_{k-1}^*)$, and, hence, $|z_k|\le 2(m_k^*-m_{k-1}^*)$. Moreover, since $m_n^*=\sup_{k\le n}|m_k|=\sum_{k=1}^n(m_k^*-m_{k-1}^*)$, it follows that $\sum_{k=1}^n|z_k|\le 2m_n^*$. Therefore,

$$\mathbb{E}\Phi(H_n^*)\le 4\,\mathbb{E}\Phi(m_n^*),\quad 1\le n\le N.$$

Since $\{|m_n|\}_{n=1}^N$ is a non negative real valued sub-martingale, we get by a generalization of Doob's maximal inequality, i.e. Proposition A.3

$$\mathbb{E}\Phi(H_n^*)\le C\,2\,\mathbb{E}\Phi(|m_n|).$$

From $m_n = M_n - M_{n-1}$, it follows that $|m_n| \le S_{n,p}(M)$, and, hence,

$$\mathbb{E}\Phi(H_n^*) \le C' \, 2 \, \mathbb{E}\Phi(S_{n,p}(M)), \quad 1 \le n \le N. \tag{A.10}$$

In the next paragraph, we will give an upper estimate of the term $\mathbb{E}\Phi(G_n^*)$. Observe, first, that $|m_k| \le 2m_{k-1}^*$ implies $|g_k| \le 4m_{k-1}^*$, $0 \le k \le N$. This means, that g_k is controlled by a $\mathcal{F}_{k-1}$-measurable random variable, and, therefore, we can apply Proposition A.8 to get a control of G^*. In particular, there exists a constant $c_{\Phi,E} < \infty$, only depending on Φ and E, such that

$$\mathbb{E}\Phi(G_n^*) \le c_{\Phi,E} \mathbb{E}\Phi(S_{n,p}(G)) + c_{\Phi,E} \mathbb{E}\Phi(m_n^*). \tag{A.11}$$

The term $\mathbb{E}\Phi(m_n^*)$ can be estimated by the generalized Doob maximal inequality A.3. So, we obtain

$$\mathbb{E}\Phi(G_n^*) \le c'_{\Phi,E} \mathbb{E}\Phi(S_{n,p}(G)) + c'_{\Phi,E} \mathbb{E}\Phi(m_n). \tag{A.12}$$

From $m_n = M_n - M_{n-1}$, it follows that $|m_n| \le S_{n,p}(M)$. It remains to investigate $\mathbb{E}\Phi(S_{n,p}(G))$. Note, that

$$\mathbb{E}\Phi(S_{n,p}(G)) \quad \le \quad c''_{E,\Phi} \, \mathbb{E}\Phi(S_{n,p}(M)) + c''_{E,\Phi} \, \mathbb{E}\Phi(S_{n,p}(H)).$$

Now, since

$$\begin{aligned} \sum_{k \le n} |z_k|^p &\le \sum_{k \le n} |m_k^* - m_{k-1}^*|^p \\ &\le |m_n^*|^{p-1} \sum_{k \le n} |m_k^* - m_{k-1}^*| \le |m_n^*|^p, \end{aligned}$$

we have by Lemma A.6 applied to $\{|z_n|^p\}_{n=1}^N$

$$\mathbb{E}\Phi(S_{n,p}(H)) \le C \, \mathbb{E}\Phi\big(m_n^*\big), \quad 1 \le n \le N.$$

Again applying Proposition A.3 leads to

$$\mathbb{E}\Phi(S_{n,p}(G)) \quad \le \quad c''_{E,\Phi} \, \mathbb{E}\Phi(S_{n,p}(M)) + c'''_{E,\Phi} \mathbb{E}\Phi\big(|m_n|\big) \le c''''_{E,\Phi} \, \mathbb{E}\Phi(S_{n,p}(M)). \tag{A.13}$$

Therefore, substituting (A.10), (A.12) and (A.13) in (A.9) gives

$$\mathbb{E}\Phi(M_n^*) \le \bar{c}_{E,\Phi} \mathbb{E}\Phi(S_{n,p}(M)), \quad 1 \le n \le N. \tag{A.14}$$

□

Appendix B. Preliminaries about convex functions

Let $\Phi : [0,\infty) \to \mathbb{R}$ be a strictly increasing convex function. By e.g. [20, Theorem A] it follows that there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, where ϕ is strictly increasing such that

$$\Phi(t) = \int_0^t \phi(s)\, ds, \quad t \ge 0.$$

To such a convex function Φ we can associate another convex function Ψ of the same type such that

$$\Psi(t) = \int_0^t \psi(s)\, ds, \quad t \ge 0.$$

and $\phi(s) = \inf\{t \geq 0 : \psi(t) \geq s\}$ and $\psi(s) = \inf\{t \geq 0 : \phi(t) \geq s\}$, $s \geq 0$. (see e.g. [20, Chapter I.15])

Particulary, the following holds (see e.g. [20, Chapter I.15, p.30])

Proposition B.1. *Let $\Phi : [0,\infty) \to \mathbb{R}$ be a strictly increasing convex function and $\Psi : [0,\infty) \to \mathbb{R}$ its conjugate. Then*

$$u\,\phi(u) = \Phi(u) + \Psi(\phi(u)), \tag{B.1}$$

$$\int_0^v t\,d\psi(t) = \Phi(\psi(v)), \tag{B.2}$$

$$u\,v \leq \Phi(u) + \Psi(v), \tag{B.3}$$

$$\Phi(ua) \leq a\Phi(a), \quad \forall 0 < a \leq 1, \tag{B.4}$$

Furthermore, we say Φ satisfies *the growth condition*, iff there exists a constant c_Φ with

$$\Phi(2\lambda) \leq c_\Phi\,\Phi(\lambda), \quad \lambda \in [0,\infty).$$

Since we will need it later, we summaries in this paragraph some facts about convex functions (see e.g. [14, Appendix] or [20]). For any increasing convex and continuous function Φ, there exists an increasing, non-negative function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that $\Phi(t) = \int_0^t \phi(s)\,ds$. We can associate to Φ a function Ψ, where $\Psi(t) = \int_0^t \psi(s)\,ds$ and $\psi(s) = \sup\{t : \phi(t) \leq s\}$ for all $s > 0$. Such a function is called conjugate to Φ in the sense of Young. If the growth condition holds, then

$$c_\Phi^* := \sup_{u>0} \frac{u\phi(u)}{\Phi(u)} \tag{B.5}$$

is finite and we get

$$\Phi(t_1 \vee t_2) \leq \Phi(t_1) + \Phi(t_2), \quad t_1, t_2 \geq 0, \tag{B.6}$$

$$\Phi(rt) \leq r^{c_\Phi^*}\Phi(t), \quad t \geq, r \geq 1, \tag{B.7}$$

$$\Psi(t) \leq (c_\Phi^* - 1)\,\Phi(\psi(t)), \quad t \geq 0. \tag{B.8}$$

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